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SUBMODEL OF ROTATIONAL MOTIONS IN GAS DYNAMICS

S. V. Khabirov

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An invariant submodel of the equations of gas dynamics constructed on a one-dimensional subalgebra consisting of the sum of operators of rotation and translation in time is studied within the framework of the SUBMODELS program. The system of equations of the submodel is brought to symmetric form. Hyperbolicity conditions for the system are derived. Group analysis is performed and an invariant solution is considered. Isobaric flows are listed. For the simplest of them, characteristics and strong discontinuities are considered. Necessary conditions for existence of solutions without singularities on the axis are derived.

1. Equations of the Submodel. The equations of gas dynamics are considered in the cylindrical coordinates t, x, r, and θ; U, V, and W are the velocity projections onto the unit vectors, and p and ρ are the pressure and density. The invariant solution [1] is written in terms of the invariants of the operator $X_7 + X_{10} = \partial_\theta + \partial_t$: $U = u$, $V = v$, $W = r(w+1)$; the functions u, v, w, p, and ρ depend on x, r, and $s = \theta - t$; therefore, on the level line of the functions, a point moves in a circle at a constant circular velocity. Substitution of this representation of the solution into the equations of gas dynamics gives the following equations of the submodel:

$$\begin{gathered} D\mathbf{u} + \rho^{-1}(p_x, p_r, r^{-2}p_s) = (0,\ r(w+1)^2,\ -2r^{-1}v(w+1)) = \mathbf{a}, \\ Dp + A\,\mathrm{div}\,\mathbf{u} = -r^{-1}Av, \qquad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = -r^{-1}\rho v. \end{gathered} \tag{1.1}$$

Here $\mathbf{u} = (u, v, w) = (u^1, u^2, u^3)$, $\mathrm{div}\,\mathbf{u} = u_x + v_r + w_s$, $D = u\partial_x + v\partial_r + w\partial_s$, $A = A(\rho, p) = \rho c^2$, $c^2 = \partial f/\partial \rho$, $p = f(\rho, S)$ is the equation of state, and S is the entropy.

System (1.1) is brought to symmetric form. For this, instead of the last equation we write the equation for the entropy $DS = 0$. The linear replacement of the velocities $v^i = b^i_k u^k$, $u^k = c^k_i v^i$, and $b^i_m c^m_k = \delta^i_k$ reduces system (1.1) to the following system for the vector function $\mathbf{q} = (v^1, v^2, v^3, p, S)^t$:

$$B^i \mathbf{q}_{x^i} = \mathbf{F}. \tag{1.2}$$

Here $x^1 = x$, $x^2 = r$, $x^3 = s$, $\mathbf{F} = (d^1, d^2, d^3, d^4, 0)^t$, $d^4 = -v^i(r^{-1}c^2_i + c^k_{i x^k})$, $d^i = b^i_k a^k + \rho b^i_{n x^l} c^l_k v^k c^n_m v^m$, $\mathbf{a} = (a^1, a^2, a^3)$,

$$B^i = \begin{bmatrix} \rho c^i_k v^k & 0 & 0 & b^1_i g^{ii} & 0 \\ 0 & \rho c^i_k v^k & 0 & b^2_i g^{ii} & 0 \\ 0 & 0 & \rho c^i_k v^k & b^3_i g^{ii} & 0 \\ c^i_1 & c^i_2 & c^i_3 & A^{-1} c^i_k v^k & 0 \\ 0 & 0 & 0 & 0 & c^i_k v^k \end{bmatrix},$$

$g^{11} = g^{22} = 1$, and $g^{33} = r^{-2}$ ($i = 1, 2, 3$).

The matrices B^i are symmetric if the following conditions are satisfied:

$$c^i_k = b^k_i g^{ii}. \tag{1.3}$$

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Hence, we obtain $|\mathbf{c}^i|^2 = g^{ii}$ and $\mathbf{c}^i \cdot \mathbf{c}^k = 0$ $(i \neq k)$, where $\mathbf{c}^i = (c_1^i, c_2^i, c_3^i)^t$. Therefore, if the direction of one of the vectors $\mathbf{c}^i$ is specified, then the matrix c_k^i is completely determined with accuracy up to the rotation around this direction.

System (1.2) and (1.3) is symmetric hyperbolic if one of the matrices B^i is positive definite. The eigenvalues B^i are calculated from the formulas

$$\lambda_1^i = u^i = c_k^i v^k, \quad \lambda_{2,3}^i = \rho u^i, \quad \lambda_{4,5}^i = \frac{1}{2}(\rho + A^{-1})u^i \pm \left(\frac{1}{4}(\rho - A^{-1})^2 (u^i)^2 + g^{ii}\right)^{1/2}.$$

This leads to the positive-definite conditions $\rho > 0$ and $u^i > c(g^{ii})^{1/2} > 0$.

The characteristics of system (1.2) and (1.3) are found from the equality

$$\det \sum_{i=1}^{3} B^i \xi^i = 0,$$

where $\xi = (\xi^1, \xi^2, \xi^3)$ is a normal vector to the characteristic surface. We obtain the triple characteristic C_0: $\sum_{i=1}^{3} u^i \xi^i = 0$, and two other characteristics are possible, which are defined by the equality

$$\left(\sum_{i=1}^{3} u^i \xi^i\right)^2 - c^2 \sum_{k=1}^{3} \left(\sum_{i=1}^{3} c_k^i \xi^i\right)^2 = 0.$$

This equality is written in matrix form as $\xi H \xi^t = 0$ [2], where $H = \mathbf{h} \otimes \mathbf{h} - G(x)$, $\mathbf{h} = (h^1, h^2, h^3)$, $h^i = c^{-1}u^i$, and $G = (g^{ik}) = \operatorname{diag}(1, 1, r^{-2})$. If the eigenvalues of the matrix H, which are all real, have different signs, there are two real characteristics $C_{\pm}$.

The equation that defines the eigenvalues has the form

$$f(\lambda) = \lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0,$$

where

$$J_1 = \operatorname{tr} H = |\mathbf{h}|^2 - \operatorname{tr} G = |\mathbf{u}|^2 c^{-2} - 2 - r^{-2};$$
$$J_2 = \operatorname{tr} H^{-1} \det H = \operatorname{tr} G^{-1} \det G - |\mathbf{h}|^2 \operatorname{tr} G + \mathbf{h} G \mathbf{h}^t = 1 + 2r^{-2} - |\mathbf{u}|^2 c^{-2}(1 + r^{-2}) + w^2 c^{-2}(r^{-2} - 1);$$
$$J_3 = \det H = \det G(-1 + \mathbf{h} G^{-1} \mathbf{h}^t) = r^{-2}(-1 + |\mathbf{u}|^2 c^{-2} + w^2 c^{-2}(r^2 - 1)).$$

According to the Routh theorem, the number of positive roots of the polynomial $f(\lambda)$ is equal to the number of sign inversions in the sequence 1, $-J_1$, $J_2 - J_3 J_1^{-1}$, and $-J_3$.

The hyperbolicity region of system (1.1) for a fixed solution is defined by the systems of inequalities

$$J_1 > 0, \quad J_2 J_1 < J_3, \quad J_3 > 0,$$
$$J_1 < 0, \quad J_2 J_1 > J_3, \quad J_3 > 0,$$
$$J_1 < 0, \quad J_2 J_1 < J_3, \quad J_3 > 0$$

if the sign inversions are one in number and by

$$J_1 > 0, \quad J_2 J_1 > J_3, \quad J_3 < 0,$$
$$J_1 > 0, \quad J_2 J_1 < J_3, \quad J_3 < 0,$$
$$J_1 < 0, \quad J_2 J_1 > J_3, \quad J_3 < 0$$

if the sign inversions are two in number

Solving these inequalities, we come to the following proposition.

Proposition 1. *System* (1.1) *is hyperbolic on a solution in the region defined by the inequality*

$$u^2 + v^2 + r^2 w^2 > c^2. \tag{1.4}$$

In physical variables, condition (1.4) becomes $U^2 + V^2 + (W - r)^2 > c^2$, from which it follows that for large r, system (1.1) is always hyperbolic even for subsonic physical velocities. For $W \simeq r$, inequality (1.4)

TABLE 1

No.	A	Augmentation operators
2	$p\varphi(p\rho^{-\gamma})$	$(\gamma-1)Z+2\gamma Y_p$
3	$p\varphi(p\rho^{-1})$	Y_p
4	$\varphi(p)$	Z
5	$p\varphi(\rho)$	$Z+2Y_p$
6	γp	Y_p, Z
8	$\varphi(\rho\,e^{-p})$	$Z-2Y_1$
9	$\varphi(\rho)$	Y_1
10	$\gamma\rho^{\gamma}$	$Y_1, (\gamma-1)Z+2\gamma Y_p$
11	ρ	Y_p, Y_1
12	1	Y_1, Z
13	0	$Z, Y_{\varphi(p)}$

is not necessarily satisfied, although the physical flow is supersonic. This distinguishes system (1.1) from the system of equations for steady flows.

2. Group Classification and Invariant Solution. System (1.1) admits the equivalence transformations $x' = a_1x$, $r' = a_1r$, $u' = a_1u$, $v' = a_1v$, $\rho' = a_2\rho$, $p' = a_1^2a_2(p+a_3)$, and $A' = a_1^2a_2A$, where a_i are parameters; the remaining variables are invariant. With an arbitrary function $A(\rho,p)$, system (1.1) admits the Abelian algebra $L_2 = \{\partial_x, \partial_s\}$, which is the kernel of admissible algebras and the normalizing factor of the subalgebra X_7+X_{10} in the algebra L_{11} of the equations of gas dynamics [1]. Augmentation of the kernel occurs in exactly the same cases as for the initial model [1] (see Table 1), except for the case $A = (5/3)p$. The numbering in Table 1 is the same as in Table 1 of [1]: $Z = x\partial_x + r\partial_r + u\partial_u + v\partial_v - 2\rho\partial_\rho$, and $Y_{\varphi(p)} = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p$, where φ is an arbitrary function. This table is a result of group classification of system (1.1) by an arbitrary element A.

Remark. Any augmentation of the kernel is a quotient algebra of the normalizer of the subalgebra X_7+X_{10} in the corresponding augmentation L_{11} on an ideal X_7+X_{10}, because, from the augmentations of Table 1 in [1], it is easy to select operators that coincide in invariant variables with the operators given in Table 1 of the present paper for the corresponding augmentations.

The optimal system of subalgebras for the kernel is obvious, because for Abelian algebras there are no nontrivial internal automorphisms. We consider an invariant solution constructed using the algebra L_2. The solution u, v, w, S, ρ depends only on r. For $v \neq 0$, five integrals for the quotient system are valid:

$$S(\rho,p) = S_0(\rho S_\rho + AS_p = 0), \quad u = u_0, \quad rv\rho = E_0, \quad r^2(1+w) = B,$$
$$E_0^2r^{-2}\rho^{-2} + I(\rho) + r^{-2}B^2 = C^2. \tag{2.1}$$

Here S_0, u_0, E_0, w_0, B, and C are constants and $I = 2\int\limits_0^\rho \rho^{-1}c^2(\rho)\,d\rho \geqslant 0$.

Proposition 2. *For an ordinary gas,* Eq. (2.1) *gives a limited two-value function $\rho(r)$ defined in the region $r \geqslant r_0 > 0$. The first branch $\rho_1 > \rho > \rho(r_0) = \rho_0$, for which the radial velocity is subsonic, increases monotonically. The second branch $0 < \rho < \rho_0$, for which the radial velocity is supersonic, decreases monotonically.*

Proof follows from the properties of the function $I(\rho)$ for an ordinary gas [3, p. 101].

In physical variables, the solution is defined by the formulas $U = u_0$, $V = E_0r^{-1}\rho^{-1}$, and $W = Br^{-1}$. It describes steady gas flow from a cylindrical nonpoint source $r \geqslant r_0$ with swirl $W \neq 0$.

For $v = 0$, the solution with three arbitrary functions

$$u(r),\ w(r),\ p(r),\ \rho = r^{-1}p'(w+1)^{-2} \tag{2.2}$$

describes steady stratified motion of particles along helical lines on the cylinders.

3. Isobaric Flows. For the submodel considered here, there is a wide class of flows with constant pressure $p = p_0$. The general solution of isobaric flows is obtained in [4]. In the present paper, we give another representation of the solution for the submodel. System (1.1) becomes overdetermined:

$$\mathbf{u}\mathbf{u}_x + v\mathbf{u}_r + w\mathbf{u}_s = (0,\ r(w+1)^2,\ -2r^{-1}v(w+1)),\ \ \operatorname{div}\mathbf{u} + r^{-1}v = 0,\ \ u\rho_x + v\rho_r + w\rho_s = 0. \tag{3.1}$$

This system admits, besides the translations ∂_x and ∂_s, two stretchings $x\partial_x + u\partial_u$ and $r\partial_r + v\partial_v$. Thus, from any exact solution with constant pressure, the indicated admissible group allows one to obtain a solution with five arbitrary constants. The general admissible group cannot be calculated until (3.1) is reduced to involution, although it is possible to indicate two other admissible operators ∂_ρ and $\rho\partial_\rho$.

System (3.1) is integrated in Lagrangian variables, one of which is the streamline parameter τ:

$$u^{-1}dx = v^{-1}dr = w^{-1}ds = d\tau,\quad x\Big|_{\tau=0} = \xi,\quad r\Big|_{\tau=0} = \eta,\quad s\Big|_{\tau=0} = \zeta. \tag{3.2}$$

Here the point (ξ, η, ζ) lies on a two-dimensional surface that is not tangent to the field (u, v, w). System (3.1) becomes

$$u_\tau = 0,\quad v_\tau = r(w+1)^2,\quad w_\tau = -2r^{-1}v(w+1),\quad \rho_\tau = 0,\quad r(u_x + w_s) + (rv)_r = 0. \tag{3.3}$$

With accuracy up to the stretchings, the solution of (3.2) and (3.3) is

$$\begin{gathered} x = \xi + \tau,\quad r^2 = \eta^2 + \tau^2 + 2\tau f,\quad \eta r\cos(s + \tau - \zeta) = \eta^2 + \tau f, \\ u = 1,\quad v = r^{-1}(\tau + f),\quad w = -1 + r^{-2}(\eta^2 - f^2)^{1/2},\quad \rho = \rho_0, \end{gathered} \tag{3.4}$$

where $f = \eta\sin(\zeta + \xi - \varphi(f - \xi))$ and $\varphi(\lambda)$ is an arbitrary function. The quantities ξ, η, ζ, and f are functions of two parameters. Any two of them can be used as parameters. The translations admitted by system (3.1) specify transformations of the similarity between the solutions. The general solution (3.4) is divided into two cases.

Case 1. ξ and η are parameters and $\zeta = 0, f(\xi, \eta)$:

$$\begin{gathered} \tau = x - \xi,\quad r\sin(s + \tau) = \tau\cos(\xi - \varphi),\quad r\cos(s + \tau) = \eta + \tau\sin(\xi - \varphi), \\ \eta\sin(\xi - \varphi) = f,\quad u = 1,\quad v = r^{-1}(\tau + f),\quad w = -1 + r^{-1}\cos(\xi - \varphi). \end{gathered}$$

For $\varphi(\lambda) = -\lambda$, we obtain an invariant ∂_x-solution.

Case 2. η and ζ are parameters and $\xi = 0, f(\zeta, \eta)$:

$$\begin{gathered} f = \eta\sin(\zeta - \varphi(f)),\quad r^2 = \eta^2 + x^2 + 2xf,\quad \eta r\cos(s + x - \zeta) = \eta^2 + xf, \\ u = 1,\quad v = r^{-1}(x + f),\quad w = -1 + r^{-2}\eta\cos(\zeta - \varphi). \end{gathered}$$

For $\varphi = 0$, the solution is defined by explicit formulas.

A simple solution is obtained from (3.4) for $f = 0$:

$$r\sin(s + x - \varphi(-\xi)) = x - \xi,\ \ u = 1,\ \ v = \sin(s + x - \varphi(-\xi)),\ \ w = -1 + r^{-1}\cos(s + x - \varphi(-\xi)).$$

If, in addition to that, $\varphi = 0$, we obtain a periodic solution which is specified everywhere except for the x axis. Using stretchings, translations, and inversions (the constants can take negative values as well), we can reduce it to a simple solution that depends on five constants:

$$\begin{gathered} u = u_0,\quad v = q_0\sin(s + u_0^{-1}x - s_0),\quad w = -1 + q_0 r^{-1}\cos(s + u_0^{-1}x - s_0), \\ p = p_0,\qquad \rho = \rho_0. \end{gathered} \tag{3.5}$$

In physical variables, the solution becomes

$$\begin{gathered} U = u_0,\quad V = q_0\sin(xu_0^{-1} + \theta - t + \theta_0),\quad W = q_0\cos(xu_0^{-1} + \theta - t + \theta_0), \\ p = p_0,\qquad \rho = \rho_0. \end{gathered} \tag{3.6}$$

This solution defines gas flow from the x axis.

4. Characteristics and strong discontinuities. Characteristic surfaces for the equations of gas dynamics can be constructed for an invariant solution of submodel (1.1) [3, p. 60].

The invariant characteristic surfaces $h(x, r, s) = \text{const}$ are defined by the submodel (1.1):

$$C_0: \quad uh_x + vh_y + wh_z = 0,$$
$$C_\pm: \quad uh_x + vh_y + wh_z \pm cq = 0, \quad q = (h_x^2 + h_r^2 + r^{-2}h_s^2)^{1/2}.$$

The bicharacteristics are defined by the equations

$$C_0: \ d_0x = u, \quad d_0r = v, \quad d_0s = w \qquad \text{(streamline)},$$
$$C_\pm: \ d_\pm x = u \pm c h_x q^{-1}, \quad d_\pm r = v \pm c h_r q^{-1}, \quad d_\pm s = w \pm c h_s r^{-2} q^{-1},$$
$$d_\pm h = \pm c(1 - r^{-2})h_s^2 q^{-1}, \quad d_\pm h_x = -u_x h_x - v_x h_r - w_x h_s \mp c_x q,$$
$$d_\pm h_r = -u_r h_x - v_r h_r - w_r h_s \pm c h_s^2 r^{-3} q^{-1} \mp c_r q, \quad d_\pm h_s = -u_s h_x - v_s h_r - w_s h_s \mp c_s q.$$

For solution (3.5), the streamline is the irregular helical line $q_0(x - x_0)\cos(s + xu_0^{-1}) = r_0u_0 \sin(s - s_0 + u_0^{-1}(x - x_0))$ on the surface of revolution formed by the hyperbola $u_0^2r^2 = q_0^2(x - x_0)^2 + 2(x - x_0)r_0u_0q_0 \sin(s_0 + x_0) + r_0^2$. In this case, the characteristic surface is defined by the equation $q_0x = u_0r \sin(u_0^{-1}x + s) + \varphi(r\cos(u_0^{-1}x + s))$. The region of existence of the characteristics $C_\pm$ is given by the inequality $r^2 - 2rq_0\cos(s + u_0^{-1}x - s_0) > C_0^2 - u_0^2 - q_0^2$.

For $r = 0$, the supersonic flow condition is obtained.

For solution (3.5), the equations of the bicharacteristics have two integrals, $u_0h_x - h_s = C_1$ and $h_r^2 + r^{-2}h_s^2 = C_2^2$, which, together with the equation of the characteristics $\lambda = s + u_0^{-1}x$, $u_0h_x + q_0 \sin\lambda h_r + (-1 + q_0r^{-1}\cos\lambda)h_s \pm c_0(h_x^2 + h_r^2 + r^{-2}h_s^2)^{1/2} = 0$, give all derivatives of the function h. Since the integrals are in involution, h is defined by the quadrature

$$h = C + C_1u_0^{-1}x + C_2\int r\cos\omega\, d\lambda + \sin\omega\, dr, \tag{4.1}$$

where the function $\omega = \omega(\lambda, r)$ satisfies the equation

$$q_0\cos(\omega - \lambda) \pm c_0[u_0^{-2}(C_1C_2^{-1} + r\cos\omega)^2 + 1]^{1/2} = 0.$$

Equation (4.1) gives the full integral of the equation for the characteristics $C_\pm$. It is used to determine characteristics for solutions [4].

The surface of a strong discontinuity in cylindrical coordinates is given by the equation $F(t, x, r, \theta) = 0$. The normal and the normal velocity are

$$\mathbf{n} = \nabla_c F|\nabla_c F|^{-1}, \qquad D_n = -F_t|\nabla_c F|^{-1}, \qquad \nabla_c = (\partial_x, \partial_r, r^{-1}\partial_\theta).$$

For an invariant surface and invariant solution the following equalities are valid:

$$F = h(x, r, s), \qquad D_n = h_s q^{-1}, \qquad \mathbf{n} = (h_x, h_r, r^{-1}h_s)q^{-1},$$
$$q = (h_x^2 + h_r^2 + r^{-2}h_s^2)^{1/2}, \qquad \omega = u_n - D_n = (uh_x + vh_r + wh_s)q^{-1},$$
$$\mathbf{n}_\sigma = \mathbf{u} - u_n\mathbf{n} = (u(1 - h_xq^{-1}), v(1 - h_rq^{-1}), w - (w + 1)q^{-1}h_s).$$

A contact discontinuity [3, p. 38] is characterized by the conditions $[p] = 0$, $u_n = D_n$, and $[\mathbf{u}_\sigma] \neq 0$. In invariants, the equalities become

$$[p] = p_2 - p_1 = 0, \quad u_ih_x + v_ih_r + w_ih_s = 0, \quad i = 1, 2. \tag{4.2}$$

The subscripts 1 and 2 denote the limiting values on the discontinuity surface on its different sides.

Proposition 3. *For solutions* (3.5), *an invariant contact discontinuity is possible only for* $u_1 = u_2 = u_0$, $[q] \neq 0$:

$$h = ru_0(q_1\cos(s + u_0^{-1}x + s_1) - q_2\cos(s + u_0^{-1}x + s_2)) + q_1q_2x\sin(s_1 - s_2) = C. \tag{4.3}$$

Proof. For solutions (3.5), equalities (4.2) are written as

$$Y_i h = u_i h_x + q_i \sin(s + s_i + u_i^{-1}x)h_r + (-1 + q_i r^{-1}\cos(s + s_i + u_i^{-1}x))h_s = 0. \tag{4.4}$$

This system should be closed, and, hence, the equation

$$\begin{aligned}[Y_1, Y_2]h &= (u_1 - u_2)(q_2 u_2^{-1}\cos(s + s_2 + u_2^{-1}x) + q_1 u_1^{-1}\cos(s + s_1 + u_1^{-1}x))h_r \\ &\quad -r^{-1}(q_2 u_2^{-1}\sin(s + s_2 + u_2^{-1}x) + q_1 u_1^{-1}\sin(s + s_1 + u_1^{-1}x))h_s = 0\end{aligned} \tag{4.5}$$

should hold by virtue of (4.4).

The invariants of Eq. (4.4) for $i = 1$ are

$$I = rq_1^{-1}\cos(s + s_1 + u_1^{-1}x), \quad J = xu_1^{-1} - rq_1^{-1}\sin(s + s_1 + u_1^{-1}x).$$

In these invariants, Eq. (4.4) for $i = 2$ becomes

$$h_I(æ(xu_1^{-1} - J) + q_2 q_1^{-1}u_2^{-1}\sin(s_2 - s_1 + æx)) + h_J(u_1^{-1} + æI + q_2 q_1^{-1}u_2^{-1}\cos(s_2 - s_1 + æx)) = 0, \tag{4.6}$$

where $æ = u_2^{-1} - u_1^{-1}$.

The functions x, $\sin(s_2 - s_1 + æx)$, and $\cos(s_2 - s_1 + æx)$ are linearly independent, and the variable x is free in (4.6). Hence, x should not enter into (4.6). This is possible only for $æ = 0$, i.e., $u_2 = u_1 = u_0$. Then, (4.5) holds identically, and solution (4.6) takes the form (4.3).

A shock wave is defined by the relations [3, p. 39]

$$[u_n] \neq 0, \quad \omega_1^2 = \frac{\rho_2}{\rho_1}\frac{p_2 - p_1}{\rho_2 - \rho_1}, \quad \omega_2^2 = \frac{\rho_1}{\rho_2}\frac{p_2 - p_1}{\rho_2 - \rho_1}, \quad H(\rho_2, p_2; \rho_1, p_1) = 0, \quad [\mathbf{u}_\sigma] = 0,$$

where H is a Hugoniot function, and u_n and $\mathbf{u}_\sigma$ are the normal velocity and the velocity component tangent to the shock-wave surface. The last vectorial equation is equivalent to the following system of equations for an invariant shock wave:

$$[u]^{-1}h_x = [v]^{-1}h_r = r^{-2}[w]^{-1}h_s = q[\omega]^{-1}. \tag{4.7}$$

This leads to the equality

$$[u]^2 + [v]^2 + r^2[w]^2 = [\omega]^2. \tag{4.8}$$

Proposition 4. *For solutions* (3.5) *a shock wave cannot exist.*

Proof. For an invariant shock wave, Eq. (4.8) takes the form

$$[\omega]^2 - [u]^2 - q_1^2 - q_2^2 + 2q_1q_2\cos([s] + x(u_2^{-1} - u_1^{-1})) = 0.$$

Hence, $u_1 = u_2 = u_0$ and $[\omega]^2 + 2q_1q_2\cos[s] = q_1^2 + q_2^2$.

From (4.7) we obtain the equalities $h_x = 0$ and

$$[\omega]^2 h_r^2 = (q_2\sin(s + s_2 + u_0^{-1}x) - q_1\sin(s + s_1 + xu_0^{-1}))(h_r^2 + r^{-2}h_s^2).$$

In the last equality, x is a free variable. Equating the x-dependent coefficients of the linearly independent functions to zero, we have $q_2 = q_1$ and $s_2 = s_1$, i.e., $[\mathbf{u}] = 0$, which contradicts the proposition.

In the same manner, it is proved that there cannot be a noninvariant shock wave.

An invariant shock wave in the form of an Archimedes screw can connect two different solutions of the form (2.2).

Proposition 5. *An invariant shock transition is possible on solutions* (2.2). In this case, *on one side of the discontinuity it is possible to specify a solution with two arbitrary functions, and on the other side, solution* (2.2) *is determined with accuracy up to a solution of an ordinary differential equation of the first order.*

Proof. Let the gas flow on both sides of an invariant shock wave $h(x, r, s) = H_0$ be defined by functions (2.2):

$$v_i = 0, \quad u_i(r), \quad w_i(r), \quad p_i(r),\ \rho_i = r^{-1}p_i'(w_i + 1)^{-2}.$$

Equalities (4.7) take the form $h_r = 0$ and $[u]^{-1}h_x = r^{-2}[w]^{-1}h_s$. Hence,

$$[u] = Cr^2[w], \tag{4.9}$$

where C is an arbitrary constant and $h = s + Cx$. Thus, the shock-wave surface is an Archimedes screw. We determine the relative velocities $\omega_i = (Cu_i + w_i)(C^2 + r^{-2})^{-1/2}$ at which equality (4.8) holds identically. From the equations of shock transition we determine

$$\rho_2 = G(p_2, p_1, r^{-1}p_1'(w_1+1)^{-2}), \qquad [\rho] = G - p_1'(w_1+1)^{-2}r^{-1},$$
$$Cu_1 + w_1 = r^{-1}(r^2C^2+1)^{1/2}(G[p])^{1/2}(\rho_1[\rho])^{-1/2},$$
$$Cu_2 + w_2 = r^{-1}(r^2C^2+1)^{1/2}(G[\rho])^{-1/2}(\rho_1[p])^{1/2}.$$

Substitution of these expressions into (4.9) gives the differential equation for p_2: $r(C^2r^2+1)p_1'(p_2'^{1/2} - (1 + w_1)(rG)^{1/2})^2 = (p_2 - p_1)(Gr(w_1+1)^2 - p_1')$. The functions $w_1(r)$ and $p_1(r)$ can be chosen arbitrarily.

On the whole, the shock-wave surface is given by one step of a helical surface. Its trace on the cylinder is bounded by helical streamlines ahead of the shock and behind it. If the angle between the velocity vector and the cylinder axis behind the shock increases, then behind the helical curve of the shock, one should place a wall bounded by the helical streamlines behind the shock and a portion of the helical curve ahead of the shock whose length is equal to the step of the helical curve of the shock. If this angle decreases, then ahead of the helical curve of the shock one should place a wall bounded by the helical streamlines ahead of the shock and a portion of the helical curve behind the jump, whose length is equal to the step of the helical curve of the jump.

5. Solutions without Singularities on the Axis. Submodel (1.1) for $r = 0$ can have a singularity. A solution without a singularity is represented as a series in nonnegative powers of the variable r. Substitution of the series into system (1.1) leads to the necessary condition for existence of such solutions. The series should have the form

$$u = \sum u_k r^k, \;\; v = r\sum v_k r^k, \;\; w = \sum w_k r^k, \;\; \rho = \sum \rho_k r^k, \;\; p = P(x) + r^2 \sum p_k r^k,$$
$$A = \sum A_k r^k, \;\; A_k = (k!)^{-1}D_r^k A(\rho,p)\big|_{r=0} = A^0_\rho \rho_k + A^0_p p_{k-2} + A^0_{\rho\rho}\rho_1\rho_{k-1} + A^0_{p\rho}\rho_1 p_{k-3} + \ldots, \tag{5.1}$$
$$A_0 = A^0 = A(\rho_0, P), \qquad A_1 = A^0_\rho \rho_1,$$

where D_r^k is the kth power of the operator of full differentiation with respect to r; summation is performed over whole $k \geqslant 0$.

The quantities with zero satisfy the equations of the principal term of the asymptotic representation

$$D_0u_0 = -\rho_0^{-1}P', \;\; D_0v_0 = -v_0^2 + (1+w_0)^2 - 2\rho_0^{-1}p_0, \;\; D_0w_0 + \rho_0^{-1}p_{0s} = -2v_0(1+w_0),$$
$$D_0\rho_0 = -A_0^{-1}\rho_0u_0P', \;\; u_{0x} + w_{0s} = -2v_0 - A_0^{-1}u_0P', \;\; D_0 = u_0\partial_x + w_0\partial_s. \tag{5.2}$$

System (5.2) is a Cauchy-type system in the variable s for $w_0 \neq 0$. The quantities $P'(x)$ and $A_0 = A(\rho_0, P)$ are arbitrary elements. The system can be subjected to group analysis. The quantities u_k, v_k, ρ_k, and p_k ($k > 0$) are determined from the linear equations

$$D_0u_k = -(u_{0x} + kv_0)u_k - u_{0s}w_k + P'\rho_0^{-2}\rho_k + g_1^k,$$
$$D_0v_k = -v_{0x}u_k - (k+2)v_0v_k + (2 + 2w_0 - v_{0s})w_k + 2p_0\rho_0^{-2}\rho_k - (k+2)\rho_0^{-1}p_k + g_2^k,$$
$$D_0w_k + \rho_0^{-1}p_{ks} = -w_{0x}u_k - 2(w_0+1)v_k - (w_{0s} + (k+2)v_0)w_k + p_{0s}\rho_0^{-2}\rho_k + g_3^k, \tag{5.3}$$
$$D_0\rho_k = (A_0^{-1}\rho_0P' - \rho_{0x})u_k - \rho_{0s}w_k + (A_0^{-1}u_0P' - A_0^{-2}\rho_0u_0P'A^0_\rho - kv_0)\rho_k + g_4^k,$$
$$u_{kx} + w_{ks} = -A_0^{-1}P'u_k - (k+2)v_k + A_0^{-2}u_0P'A^0_\rho\rho_k + g_5^k,$$

where g_i^k are expressed in terms of u_l, v_l, w_l, ρ_l, and p_{l-1}, $l = 0, \ldots, (k-1)$. For $k = 1$, we obtain the homogeneous linear system $g_i^1 = 0$, $i = 1, \ldots, 5$.

System (5.3) is a Cauchy-type system in the variable s for $w_0 \neq 0$. Thus, the formal series (5.1) can be constructed. This form of series is necessary for existence of a solution without a singularity on the axis.

The algebra admitted by (5.2) is sometimes extended to the variables of systems (5.3). For example, for $P'' = 0$, the algebra of system (1.1) is admitted, and, hence, invariant solutions for Eqs. (5.2) and (5.3) can be constructed.

In the case $w_0 = 0$, system (5.2) should be tested for compatibility. There are two integrals $A(\rho_0, P)\rho_{0x} = -\rho_0 P' \implies S(\rho_0, P) = C(s) \implies \rho_0 = g(P, C)$ and $u_0^2 + I(P, C) = E^2(s)$, where $I = 2\int g^{-1}(P, C)\, dP$, which give the form of the solution

$$2p_0 = \rho_0\Big[1 - \frac{1}{2}P''(\rho_0^{-1} - u_0^2 A_0^{-1}) - \frac{1}{2}P'^2\Big(\frac{3}{2}\rho_0^{-2}u_0^{-2} - \rho_0^{-1}A_0^{-1} + A_0^{-3}u_0^2\Big(\frac{1}{2}A_0^2 + A^0 A_p^0 + \rho_0 A_\rho^0\Big)\Big)\Big],$$

$$2v_0 = P'(\rho_0^{-1}u_0^{-1} - u_0 A_0^{-1}).$$

It remains to satisfy the compatibility equation $u_0 p_{0s} + (1 - u_0^2 \rho_0 A_0^{-1})P' = 0$.

For an arbitrary function $A(\rho_0, P)$, this is possible only for $P' = 0$. Hence, P and C are constants, $\rho_0 = C_0$ is a constant, $p_0 = (1/2)C_0$, $v_0 = 0$, and $u_0 = E(s)$ is an arbitrary function. Systems (5.3) are thus compatible.

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